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ON THE SPECTRUM OF THE GENERALIZED DIFFERENCE OPERATOR Δ_α OVER THE SEQUENCE SPACE $l_p\ (1 \le p < \infty)$

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İn the present work we introduce the generalized difference operator Δ_α on the sequence space $l_p\ (1 \le p < \infty)$. The operator Δ_α on l_p is defined by $\Delta_\alpha x = (a_n x_n - a_{n-1} x_{n-1})_{n=0}^{\infty}$ with $x_{-1} = a_{-1} = 0$, $a_n = a(= const)$ for $n \ge n_0$, n_0 -fixed natural number. In this work, it is shown that $\Delta_\alpha : l_p \to l_p$ is a linear bounded operator and the spectrum of the operator Δ_α over the sequence space l_p has been determined.

Keywords: spectrum, difference operator, sequence space l_p.

1. Preliminaries, background and notation

By $B(X)$, we denote the set of all linear bounded operators acting from Banach space X into itself. Let $X \ne \{\theta\}$ (θ is a zero element of X) be a complex normed space and consider a linear operator $T : D(T) \to X$ with domain $D(T) \subset X$. With X, we associate the operator

$$T_\lambda = T - \lambda I$$

where λ is a complex number and I is the idendity operator on $D(T)$. If T_λ has an inverse, we denote it by T_λ^{-1}, that is

$$T_\lambda^{-1} = (T - \lambda I)^{-1},$$

and call it the resolvent operator of T. By a regular value λ of T is meant a complex number such that T_λ^{-1} exists, is bounded and is defined on a set which is dense in X.

The resolvent set $\rho(T, X)$ of T is the set of all regular values λ of T. Its complement $\sigma(T, X) = C \setminus \rho(T, X)$ in the complex plane C is called the spectrum of T.

By w, we denote the space of all real or complex valued sequences. Any vector subspace of w is called a sequence space. We write $l_p\ (1 \le p < \infty)$ for the space consisting of all p – absolutely summable sequences, i.e.

$$l_p = \left\{ x = (x_n); \sum_{n=0}^{\infty} |x_n|^p < \infty \right\}$$

with norm

$$\|x\|_{l_p} = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p}$$

and l_∞, c_0 and c for the sequence spaces of all bounded, null and convergent, respectively, and bv_p $(1 \leq p < \infty)$ is the space consisting of all sequences (x_n) such that $(x_n - x_{n+1})$ is in l_p.

Let α и β be any sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in N = \{0,1,2,...\}$. Then we say that A defines a matrix mapping from α into β, and denote it by writing $A: \alpha \to \beta$, if for every sequence $x = (x_k) \in \alpha$ the sequence $Ax = \{(Ax)_n\}$, the A - transform of x, is in β; where

$$(Ax)_n = \sum_k a_{nk} x_k, (n \in N).$$

For simplicity in notation, here and in what follows, the summation without limits runs from o to ∞.

We introduce the operator Δ_α on some sequence space $\lambda \in w$ as follows,

$\Delta_\alpha : \lambda \to \lambda$ is defined by

$$\Delta_\alpha x = \Delta_a(x_n) = (a_n x_n - a_{n-1} x_{n-1})_{n=0}^{\infty}$$

with $x_{-1} = a_{-1} = 0$, $a_n = a(= const)$ for $n \geq n_0$, n_0 -fixed natural number.

The fine spectrum of the difference operator Δ over the sequence spaces c_0 and c are determined by Akhmedov and Basar [1] and B.Altay and F.Basar [2].B.de Malafosse [3] computed the spectrum of the difference operator on s_r. Akhmedov and Basar have determined the fine spectrum of the difference operator on the space bv_p $(1 \leq p < \infty)$.

Note that the sequence space bv_p was introduced and studied by Altay and Basar [4]. The continuous dual of bv_p determined by Akhmedov and Basar [5]. In this work they have determined the norm and fine spectra of the operator Δ on the space bv_p . Srivastava and Kumar [6] have recently introduced the operator Δ_v on the sequence space c_0:

$$\Delta_v : c_0 \to c_0,$$

$$\Delta_v x = \Delta_v(x_n) = (v_n x_n - v_{n-1} x_{n-1})_{n=0}^{\infty}$$

with $x_{-1}=0$, $v=(v_n)$ be either constant or strictry decreasing sequence of positive real numbers satisfying

$$\lim_{n\to\infty} v_n = L > 0 \text{ and}$$
$$\sup_n v_n \le 2L.$$

They have determined the fine spectrum of the operator Δ_v on the space c_0. Note that the results of $[6]$ for the constant v_k one can get very casy from work $[2]$.

In this work, our purpose is to show that $\Delta_a \in B(l_p)$ and to investigate the fine spectrum of the operator Δ_a on l_p $(1 \le p < \infty)$.

2.The fine spectrum of the operator Δ_a over the sequence space l_p $(1 \le p < \infty)$

In this section we shall show that $\Delta_a \in B(l_p)$ and examine the fine spectrum of generalized difference operator Δ_a on space l_p $(1 \le p < \infty)$.

Theorem 2.1. $\Delta_\alpha : l_p \to l_p$ a linear bounded operator and

$$\|\Delta_a\|_{l_p} \le 2d_a, \tag{1}$$

where $d_a = \max\left(\max_{0\le j\le n_0-1}|a_j|, |a|\right)$.

Proof. If $x=(x_n)\in l_p$, one can calculate that

$$\|\Delta_a x\|_{l_p}^p = \sum_{n=0}^{\infty}\left|a_n x_n - a_{n-1}x_{n-1}\right|^p.$$

Therefore, we obtain that

$$\|\Delta_a x\|_{l_p}^p \le |a_0|^p|x_0|^p + 2^{p-1}\left(|a_1|^p|x_1|^p + |a_0|^p|x_0|^p\right)+\ldots$$
$$\ldots + 2^{p-1}\left(\left|a_{n_0-1}\right|^p\left|x_{n_0-1}\right|^p + \left|a_{n_0-2}\right|^p\left|x_{n_0-2}\right|^p\right)+$$
$$+2^{p-1}\left(|a|^p\left|x_{n_0}\right|^p + \left|a_{n_0-1}\right|^p\left|x_{n_0-1}\right|^p\right) + 2^{p-1}|a|^p\sum_k\left(\left|x_{n_0+k}\right|^p + \left|x_{n_0+k-1}\right|^p\right) \le$$
$$\le 2^p \max\left(|a_0|^p, |a_1|^p, \ldots, \left|a_{n_0-1}\right|^p, |a|^p \cdot \|x\|_p^p\right).$$

This completes the proof.

Theorem 2.2.

$$\sigma(\Delta_a, l_p) = \left\{\lambda : \left|1-\frac{\lambda}{a}\right| \le 1\right\} \bigcup \{a_0, a_1, \ldots, a_{n_0-1}\}.$$

Proof. Let $y=(y_k)\in l_p$ and solve the equation $(\Delta_a-\lambda I)x=y$, where $y=(y_n)\in l_p$. One can calculate that

$$x_0=\frac{1}{a_0-\lambda}y_0,$$

$$x_1=\frac{a_0}{(a_0-\lambda)(a_1-\lambda)}y_0+\frac{1}{a_1-\lambda}y_1,$$

$$x_1=\frac{a_0a_1}{(a_0-\lambda)(a_1-\lambda)}y_0+\frac{a_1}{(a_0-\lambda)(a_1-\lambda)}y_1+\frac{1}{a_2-\lambda}y_2,$$

....

$$x_{n_0}=\frac{a_0a_1..a_{n_0-1}}{(a_0-\lambda)(a_1-\lambda)...(a_{n_0-1}-\lambda)(a-\lambda)}y_0+\frac{a_0a_1..a_{n_0-1}}{(a_1-\lambda)(a_2-\lambda)...(a_{n_0-1}-\lambda)(a-\lambda)}y_1+$$

$$+...+\frac{a_{n_0-1}}{(a_{n_0-1}-\lambda)(a-\lambda)}y_{n_0-1}+\frac{1}{a-\lambda}y_{n_0},$$

$$x_{n_0+1}=\frac{a_0a_1..a_{n_0-1}\cdot a}{(a_0-\lambda)(a_1-\lambda)...(a_{n_0-1}-\lambda)(a-\lambda)^2}y_0+\frac{a_0a_1..a_{n_0-1}\cdot a}{(a_1-\lambda)(a_2-\lambda)...(a_{n_0-1}-\lambda)(a-\lambda)^2}y_1+$$

$$+...+\frac{a}{(a-\lambda)^2}y_{n_0}+\frac{1}{a-\lambda}y_{n_0+1},$$

$$x_{n_0+k}=\frac{a_0a_1...a_{n_0-1}\cdot a^k}{(a_0-\lambda)(a_1-\lambda)...(a_{n_0-1}-\lambda)(a-\lambda)^{k+1}}y_0+\frac{a_0a_1...a_{n_0-1}\cdot a^k}{(a_1-\lambda)...(a_{n_0-1}-\lambda)(a-\lambda)^{k+1}}y_1+$$

$$+...+\frac{a^k}{(a-\lambda)}y_{n_0-1}+\frac{1}{a-\lambda}y_{n_0+k},$$

In order to show that $(x_n)\in l_p$, it is sufficient to prove that

$$\sum_{k=1}^{\infty}\left|x_{n_0+k}\right|^p<\infty \tag{2}$$

From the representation of x_{n_0+k} we can get

$$\sum_k\left|x_{n_0+k}\right|^p\le\sum_k(n_0+k+1)^{p-1}\Bigg(\frac{\left|a_0...a_{n_0-1}\right|^p\cdot|a|^{kp}}{|a_0-\lambda|^p...\left|a_{n_0-1}-\lambda\right|^p\cdot|a-\lambda|^{(k+1)p}}|y_0|^p+$$

$$+\frac{\left|a_1...a_{n_0-1}\right|^p\cdot|a|^{kp}}{|a_1-\lambda|^p...\left|a_{n_0-1}-\lambda\right|^p\cdot|a-\lambda|^{(k+1)p}}|y_1|^p+...+\frac{|a|^{kp}}{|a-\lambda|^{(k+1)p}}\left|y_{n_0+k-1}\right|^p+\frac{1}{|a-\lambda|^p}\cdot\left|y_{n_0+k}\right|^p\Bigg)$$

Using the method of [1] in last inequality one can show that the series (2) converges and also $(x_n) \in l_p$, if $\left|1-\frac{\lambda}{a}\right| > 1$ or $|a-\lambda| > |a|$ and $\lambda \neq a_j$, $j = 0,1,\ldots,n_0 - 1$.

That is to say that the inclusion

$$\sigma(\Delta_a, l_p) \subset \left\{\lambda : \left|1 - \frac{\lambda}{a}\right| \le 1\right\} \cup \{a_0, a_1, \ldots, a_{n_0-1}\} \qquad (3)$$

holds. Conversely, suppose that $\lambda \notin \sigma(\Delta_a, l_p)$. Then $(\Delta_a - \lambda I)^{-1} \in B(l_p)$. Since the $(\Delta_a - \lambda I)^{-1}$ -transform of the unit sequence $e_1 = (1,0,0,\ldots,)$ is in l_p, we have $|\lambda - a| > |a|$ and $\lambda \neq a_j, j = 0,1,,\ldots,n_0 - 1$ .This lead us to the inclusion

$$\{\lambda : |a-\lambda| \le |a|\} \bigcup \{\lambda \neq a_j, j = 0,1,\ldots,n_0 - 1\} \subset \sigma(\Delta_a, l_p) \qquad (4)$$

Combining the inclusions (3)and (4) we obtain the desired result. This completes the proof.

Now in order to find the norm of the operator Δ_a acting on the sequence space l_p, we wish to define the spectral radius of a bounded linear operator, which is needed in the proof .

The spectral radius $r(A)$ of an operator $A \in B(X)$ on complex Banach space X is the radius of the smallest closed disk centered at the origin of the complex $\lambda -$ plane,i.e.,

$$r(A) = \sup|\lambda|,$$
$$\lambda \in \sigma(A, X)$$

and containing $\sigma(A, X)$, (see [7], p. 378).

It is obvious that

$$r(A) \le \|A\|. \qquad (4)$$

Theorem 2.3.

$$\|\Delta_a\|_{(l_p, l_p)} = 2d_a$$

Proof. From theorem 2.2 it follows that $r(\Delta_a) = 2d_a$

Besides this, by (1) and (4) we obtain that

$$\|\Delta_A\|_{(l_p, l_p)} = 2,$$

as desired. This step completes the proof.

REFERENCES

1. A.M.Akhmedov and F.Basar. On the fine spectrum of the difference operator Δ over the sequence space $l_p(1 \le p < \infty)$//Demonstratio Math., v. 39, 2006, №3, p.585-595.
2. B.Altay and F.Basar. On the fine spectrum of the difference operator on co and c // Unform.Sci. 2004, 168, p.217-224.

3. B.de Malafosse.Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ //Hokkaido Math., t.31, 2002, pp.283-299.
4. F.Basar and B.Altay. On the space of sequence of p-bounded variation and related matrix mapping//Ukrainian Math., t. 55(1), 2003, p.136-147.
5. A.M.Akhmedov and F.Basar. The fine spectrum of the difference operator Δ over the sequence space $bv_p(1 \le p < \infty)$//Acta Math., Sinica, v.23, 2007, №10, pp.1757-1768.
6. P.D.Srivastava and S.Kumar. On the fine spectrum of the generalized difference operator Δ_v over the sequence space C_0 //Comm.in Math, Analysis, v.6, 2009, №1, pp.8-21.
7.E. Kreyszig. Introductory Functional Analysis with Applications. John Wiley & Sons Inc. New York, 1978, p.610.

$l_p(1 \le p < \infty)$ FƏZASINDA ÜMUMİLƏŞMİŞ Δ_a FƏRQ OPERATORUNUN SPEKTRİ HAQQINDA

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XÜLASƏ

İşdə $l_p(1 \le p < \infty)$ ardıcıllıqlar fəzasında təsir edən yeni Δ_a ümumiləşmiş fərq operatoru daxil edilmişdir. Δ_a operatoru aşağıdakı kimi təsir dir:

$$\Delta_a x = (a_n x_n - a_{n-1}x_{n-1})_{n=0}^{\infty}, x = (x_n), x_{-1} = a_{-1} = 0, a_n = a(= const.)$$

$n \ge n_0$, n_0 -müəyyən natural ədəddir.

İşdə $\Delta_a : l_p \to l_p$ operatorunun xətti məhdud olması və spektri təyin olunmuşdur.

О СПЕКТРЕ ОБОБЩЕНННОГО ДИФФЕРЕНЦИАЛЬНОГО ОПЕРАТОРА Δ_a В ПРОСТРАНСТВЕ $l_p(1 \le p < \infty)$

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РЕЗЮМЕ

В работе вводится новый оператор Δ_a, действующий в пространстве $l_p(1 \le p < \infty)$. Оператор Δ_a в пространстве l_p определяется следующим образом:
$\Delta_a x = (a_n x_n - a_{n-1}x_{n-1})_{n=0}^{\infty}$, $x = (x_n), x_{-1} = a_{-1} = 0, a_n = a(= const)$ для $n \ge n_0$, n_0 - фиксированное натуральное число. В работе доказывается, что $\Delta_a : l_p \to l_p$ является линейным ограниченным оператором. Также находится спектр оператора Δ_a в пространстве l_p.